$$-\frac{d}{dx}\left(\frac{\mu' A'}{r}\,dydz\right)dx = \mu' A'\,\frac{x}{r^3}\,dxdydz.$$

At every point of the medium the current is then made up of two parts, $4\pi\mu' A'$ and another which we may call $\mu'\alpha'$. So that we may write for the component of the current

$$a' = \mu'\,(\alpha' + 4\pi A').$$

We can deduce a similar equation for magnetism, indeed, one that is already known, except that we shall have what I call the magneto-motive force (similar to coercive force as used by some) in the place of the magnetization. We may then write out the equations as follows:

$$a = \mu\,(\alpha + 4\pi A)$$
$$b = \mu\,(\beta + 4\pi B)$$
$$c = \mu\,(\gamma + 4\pi C).$$

The quantities α, β, and γ are evidently the components of the magnetic *force* in the direction of the axes, and a, b, c of the magnetic induction.

$$a' = \mu'\,(\alpha' + 4\pi A')$$
$$b' = \mu'\,(\beta' + 4\pi B')$$
$$c' = \mu'\,(\gamma' + 4\pi C').$$

The quantities α', β', and γ' are evidently the components of the electric force in the direction of the axes, and a', b', c' of the current.

The quantities a, b, c, a', b', c', and α, β, γ, α', β', γ' differ essentially from each other, inasmuch as the first indicate cycles, but the second are necessarily acyclic. The quantities indicated by the Greek letters can always be obtained from acyclic potentials.

In this case we have, taking one equation as an illustration,

$$\alpha = \frac{dV}{dx}$$

$$\mu\alpha = -\,\Delta^2 L_1 + \frac{dJ}{dx},$$

$$\text{or}\quad \Delta^2 L_1 = \frac{d}{dx}\,(J - \mu V).$$

Hence, by changing the values of χ and χ' from their former values to

$$\chi = \frac{1}{\mu}\iiint (J - \mu V)\,\frac{1}{r}\,dxdydz, \qquad \chi' = \frac{1}{\mu'}\iiint (J' - \mu' V')\,\frac{1}{r}\,dxdydz,$$

we can write

$$L = \mu \left\{ \iiint \frac{A}{r}\, dxdydz - \frac{d\chi}{dx} \right\} \qquad L' = \mu' \left\{ \iiint \frac{A'}{r}\, dxdydz - \frac{d\chi'}{dx} \right\}$$

$$M = \mu \left\{ \iiint \frac{B}{r}\, dxdydz - \frac{d\chi}{dy} \right\} \qquad M' = \mu' \left\{ \iiint \frac{B'}{r}\, dxdydz - \frac{d\chi'}{dy} \right\}$$

$$N = \mu \left\{ \iiint \frac{C}{r}\, dxdydz - \frac{d\chi}{dz} \right\} \qquad N' = \mu' \left\{ \iiint \frac{C'}{r}\, dxdydz - \frac{d\chi'}{dz} \right\}$$

$$F = \frac{\mu}{4\pi} \iiint \left\{ p\left(\frac{dB}{dx} - \frac{dA}{dy}\right) + B\frac{dp}{dx} - A\frac{dp}{dy} \right\} dxdydz \qquad F' = \frac{\mu'}{4\pi} \iiint \left\{ p\left(\frac{dB'}{dx} - \frac{dA'}{dy}\right) + B'\frac{dp}{dx} - A'\frac{dp}{dy} \right\} dxdydz$$

$$G = \frac{\mu}{4\pi} \iiint \left\{ p\left(\frac{dA}{dz} - \frac{dC}{dx}\right) + A\frac{dp}{dz} - C\frac{dp}{dx} \right\} dxdydz \qquad G' = \frac{\mu'}{4\pi} \iiint \left\{ p\left(\frac{dA'}{dz} - \frac{dC'}{dx}\right) + A'\frac{dp}{dz} - C'\frac{dp}{dx} \right\} dxdydz$$

$$H = \frac{\mu}{4\pi} \iiint \left\{ p\left(\frac{dB}{dx} - \frac{dA}{dy}\right) + B\frac{dp}{dx} - A\frac{dp}{dy} \right\} dxdydz. \qquad H' = \frac{\mu'}{4\pi} \iiint \left\{ p\left(\frac{dB'}{dx} - \frac{dA'}{dy}\right) + B'\frac{dp}{dx} - A'\frac{dp}{dy} \right\} dxdydz.$$

These equations have thus been put into a form depending only on the electro-motive and magneto-motive forces, an extremely important modification. All the equations which we have so far obtained are entirely independent of any relation between electricity and magnetism; and they are perfectly true, whether that relation is the one hitherto known or also includes the relation just discovered. The equations on the left hand are all more or less known, but the similar equations on the right are for the most part new. I shall first give the relations between the two systems of equations on the old theory, so as to contrast them with the new. We remark, in the first place, that it is supposed in these equations that the electric currents are due entirely to electro-motive forces, and the magnetism to magneto-motive forces. But where closed electric currents exist, magnetism must always exist; and thus to calculate the magnetic effect by these equations, we must replace each electric current by its equivalent magneto-motive shell. It is readily seen that this will be accomplished by the addition of terms depending on the current, and hence we may write

$$\frac{dc}{dy} - \frac{db}{dz} = 4\pi\mu \left\{ a' + \frac{dC}{dy} - \frac{dB}{dz} \right\}$$

$$\frac{da}{dz} - \frac{dc}{dx} = 4\pi\mu \left\{ b' + \frac{dA}{dz} - \frac{dC}{dx} \right\}$$

$$\frac{db}{dx} - \frac{da}{dy} = 4\pi\mu \left\{ c' + \frac{dB}{dx} - \frac{dA}{dy} \right\}.$$

Equations similar to this were first given by Sir William Thomson, except

that his applied to magnetic force rather than magnetic induction, and so did not contain the terms in A, B, and C. They would thus be

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi a'$$

$$\frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 4\pi b'$$

$$\frac{d\beta}{dx} - \frac{d\alpha}{dy} = 4\pi c'.$$

As equations of this form exactly define α, β, and γ, in terms of A', B', and C', we must have

$$4\pi F' = \alpha, \qquad 4\pi G' = \beta, \qquad 4\pi H' = \gamma$$

and

$$4\pi\mu\left(L' + \mu'\frac{d\chi'}{dx}\right) = F - \mu\iiint\left\{\left(\frac{d\,(pC)}{dy} - \frac{d\,(pB)}{dz}\right)\right\}dxdydz$$

$$4\pi\mu\left(M' + \mu'\frac{d\chi'}{dy}\right) = G - \mu\iiint\left\{p\left(\frac{d\,(pA)}{dz} - \frac{d\,(pC)}{dx}\right)dxdydz\right.$$

$$4\pi\mu\left(N' + \mu'\frac{d\chi'}{dz}\right) = H - \mu\iiint\left\{p\left(\frac{d\,(pB)}{dx} - \frac{d\,(pA)}{dy}\right)dxdydz.\right.$$

Where there is no magneto-motive force (no permanent magnets), and we choose L', M', and N', so as to satisfy the equation of continuity, we have

$$4\pi\mu L' = F, \qquad 4\pi\mu' M' = G, \qquad 4\pi\mu' N' = H.$$

From the almost perfect reciprocity between electric currents and magnetic induction which I have developed in this paper, we might be led to expect that similar expressions might be found for the other case. But this is not so according to our usually conceived ideas, for lines of magnetic force must always exist around a wire carrying a current, but currents need not necessarily exist around magnets.

As we know that all the electric currents and all the magnetic force must depend only upon the terms containing A', B', and C', and A, B, and C, therefore we may select a series of terms as follows to represent the final result:

$$L = \mu\left\{\iiint Apdxdydz - \frac{d\chi}{dx}\right\}$$

$$M = \mu\left\{\iiint Bpdxdydz - \frac{d\chi}{dy}\right\}$$

$$N = \mu\left\{\iiint Cpdxdydz - \frac{d\chi}{dz}\right\}$$

$$F = \frac{dN}{dy} - \frac{dM}{dz} + 4\pi\mu\mu' \left\{ \iiint A'pdxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = \frac{dL}{dz} - \frac{dN}{dx} + 4\pi\mu\mu' \left\{ \iiint B'pdxdydz - \frac{d\chi'}{dy} \right\}$$

$$H = \frac{dM}{dx} - \frac{dL}{dy} + 4\pi\mu\mu' \left\{ \iiint C'pdxdydz - \frac{d\chi'}{dz} \right\}$$

$$a = \frac{dH}{dy} - \frac{dG}{dz} \qquad 4\pi\mu a' = \frac{dc}{dy} - \frac{db}{dz} - 4\pi\mu \left\{ \frac{dC}{dy} - \frac{dB}{dz} \right\}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \qquad 4\pi\mu b' = \frac{da}{dz} - \frac{dc}{dx} - 4\pi\mu \left\{ \frac{dA}{dz} - \frac{dC}{dx} \right\}$$

$$c = \frac{dG}{dz} - \frac{dF}{dy} \qquad 4\pi\mu c' = \frac{db}{dx} - \frac{da}{dy} - 4\pi\mu \left\{ \frac{dB}{dx} - \frac{dA}{dy} \right\}.$$

We know that every magnetic action can be represented by electric currents, therefore let us suppose the permanent magnets to be replaced by their equivalent electric currents. We then have simply

$$F = 4\pi\mu\mu' \left\{ \iiint A'pdxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = 4\pi\mu\mu' \left\{ \iiint B'pdxdydz - \frac{d\chi'}{dy} \right\}$$

$$H = 4\pi\mu\mu' \left\{ \iiint C'pdxdydz - \frac{d\chi'}{dz} \right\}$$

$$a = \frac{dH}{dy} - \frac{dG}{dz} \qquad 4\pi\mu a' = \frac{dc}{dy} - \frac{db}{dz}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \qquad 4\pi\mu b' = \frac{da}{dz} - \frac{dc}{dx}$$

$$c = \frac{dG}{dx} - \frac{dF}{dy} \qquad 4\pi\mu c' = \frac{db}{dx} - \frac{da}{dy}.$$

These equations at first sight seem to be similar to those already in use, and which are given in Maxwell's "Treatise," Art. 616. But if we examine them further, we shall see that the values of F, G, and H are given in terms of A', B', and C', rather than in terms of a', b', and c', and the introduction of these electro-motive forces together with the idea of magneto-motive forces, are the principal new points in the above theory, and are very necessary to the further extension of the theory to include the new electro-magnetic action.

III. *Physical Theory of Magnetic Attraction and Repulsion.*

The form in which the formulæ are thus thrown by expressing them in terms of the electro-motive forces has suggested to me the following physical theory of magnetism, which I give here before proceeding further.

They evidently point to the existence of a fluid filling all space, the components of whose velocity are $-F$, $-G$, and $-H$, the vortex filaments of which are the lines of magnetic induction and the vortices of the vortices, or the relative motions, of which constitute the electric currents. The motion of this fluid, we see, depends upon quantities of two kinds. The first kind includes the terms containing A', B', and C', or the electro-motive forces; and these forces must be conceived existing at various points, and tending to propel and rotate the fluid in certain directions. The second series of terms containing χ' are those portions of the components of the fluid motion which it is possible to produce by the existence in the fluid of points at which the fluid is being constantly generated or destroyed, or by moving bodies within the fluid. In other words, they indicate the motion of translation of the fluid. The calculation of χ' is thus necessary to make F, G, and H satisfy the equation of continuity, and thus to represent the components of fluid motion; but as they disappear from the succeeding equations, they are not necessary for the calculation of either the magnetic action or the electric currents.

The disappearance of χ' from the succeeding equations gives the theorem in vortex motion that no vortices can be produced in a fluid by external forces, and in electricity the theorem, which I have before demonstrated, *that electric currents in a continuous medium can never produce magnetic action unless they are closed.*

We have thus to conceive of a medium, the ordinary motions of which do not produce magnetic action or constitute electric currents, but the vortices of which are the magnetic lines of force, and the *relative* motions the electric currents.

The idea of such an extended conducting medium with its electric currents has thus lead us to the idea of a fluid filling all space, of whose ordinary motions we are unconscious, and which the earth may be whirling through with its full velocity without our being conscious of it, seeing that no magnetic action would be thus produced. But does not this give a possible explanation of the magnetism of the earth, seeing that the earth's rotation might produce rotation of this fluid which would be magnetic action? We might call this fluid *ether* if we pleased; but I do not wish this undeveloped theory to be confounded with the very crude, unscientific, and altogether untenable theory of Edlund; for

the fluid which I conceive of has none of the properties of Edlund's so-called ether.

Whether this theory could be adapted to explain electrostatic phenomena, and hence the propagation of light on Maxwell's theory, I have not yet determined. But it is to be noted that Maxwell's equations are in terms of F, G, and H, and indicate that the waves of light are waves in the fluid of which I conceive. *Could a property be added to this fluid to explain electrostatic phenomena, the fluid would therefore be identical in properties with the light ether.* This property is that on which the so-called electric displacement of Maxwell depends. Maxwell has worked up a theory similar to this in some respects, in the Phil. Mag. for 1861 and 1862; which, however, required a very artificial constitution of the ether, as he could not conceive of any method of making vortex rings in a perfect fluid. But I will show further on that my idea of electro-motive force gets over this difficulty, and allows us to explain all magnetic attractions and repulsions by the motion of a *perfect* fluid.

To get the true motion of the fluid we must be able to compute χ', although this is unnecessary as far as the magnetic action or the electric currents are concerned. To accomplish this, it is best to investigate the value of χ' for an element of electro-motive force, and then it will always appear in other cases as a definite integral. The values for an element of electro-motive force, $A'dxdydz$, at the origin are

$$F = 4\pi\mu\mu' \left\{ \frac{A'}{r}\, dxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = -\, 4\pi\mu\mu' \,\frac{d\chi'}{dy}$$

$$H = -\, 4\pi\mu\mu' \,\frac{d\chi'}{dz}.$$

The equation of continuity gives therefore

$$4\pi\mu\mu' \left\{ A' \frac{x - x'}{r^3}\, dxdydz + \Delta^2\chi' \right\} = 0,$$

whence
$$\chi' = \frac{A'dxdydz}{4\pi} \iiint \frac{x - x'}{r^3 R}\, dxdydz.$$

But as this integration is very difficult, I make use of the theorem

$$\Delta^2 r = \frac{2}{r},$$

which gives by differentiation

$$\Delta^2\left(\frac{x-x'}{r}\right) = -2\,\frac{x-x'}{r^3},$$

and hence

$$\chi' = \frac{A'}{2}\,\frac{x-x'}{r}\,dxdydz.$$

If we also had at the given point electro-motive forces B' and C', then we should have finally

$$F = 4\pi\mu\mu'\left\{\frac{A'}{r}\,dxdydz - \frac{d\chi'}{dx}\right\}$$

$$G = 4\pi\mu\mu'\left\{\frac{B'}{r}\,dxdydz - \frac{d\chi'}{dy}\right\}$$

$$H = 4\pi\mu\mu'\left\{\frac{C'}{r}\,dxdydz - \frac{d\chi'}{dz}\right\},$$

where in general

$$\chi' = \frac{1}{2r}\left\{A'(x-x') + B'(y-y') + C'(z-z')\right\} dxdydz.$$

χ' will evidently disappear when A', B', and C' satisfy the equation of continuity. This happens when the electro-motive force forms circuits which are perfectly closed.

The integrals of these expressions taken throughout space will give the values for any case.

As the motion of the imaginary fluid is rotational, it is at first sight very difficult to conceive how this rotation can be produced. For Helmholtz has shown that vortex filaments can neither be created nor destroyed by any forces acting on the fluid from without. Our problem is to conceive how an electro-motive force of intensity A', acting within an element, can fill the whole space with vortex rings, whose intensities are given by the equations.

At first I was puzzled, but finally conceived the following solution. Let the nature of electro-motive force be such that it tends to form vortex rings immediately around itself, not by action at a distance, but by direct action on the fluid in the immediate vicinity. The first ring will move forward, another one will form, and so on until all space is filled with them, when there will be equilibrium. I have not yet attempted the whole dynamics of the subject.

Magnetic attractions can be explained as follows. Conceive the fluid in a tube to be rotating around its axis with a certain velocity, and suppose the ends of the tube to be closed with movable pistons. Then, if the pistons are left

free, there will be a centrifugal force against the sides of the tube proportional to the square of the velocity of angular rotation. If the walls are flexible and the pistons immovable, then there will be a force tending to press the pistons in, and proportional also to the square of the velocity.

According to our theory the magnetic force is the velocity of rotation, and so we have in the medium a tension along the lines of force, and a pressure at right angles to them.

To satisfy the conservation of energy and the equation of continuity of the fluid, the work done on the pistons or on the envelope, together with the volume passed over, must be the same; and hence the pressure and the tension must be the same numerically.

Now Maxwell has shown that all magnetic attractions or repulsions can be accounted for by a tension along the lines of magnetic force, together with an equal pressure at right angles to them. *Hence the motion of such a fluid as we have been considering will account for all magnetic action either of magnets or of electric currents.*

Again, Sir William Thomson has shown, in his analogy of magnetism to fluid motion, that two points in a liquid at which the liquid is generated, so that there are constant currents out from them, *attract* each other. In other words, his analogy is not perfect; for, although the lines of force are of the same shape as the stream lines, yet like poles would attract, and unlike repel. But the theory of this paper, which considers lines of force as vortex filaments in a *perfect* fluid, gives a true analogy, which we have seen above accounts for all magnetic action. But Sir William Thomson's principle does not explain such action, for it would vanish for a closed circuit; and would thus give that term which has an arbitrary value in Ampère's theory, and which likewise vanishes for a closed circuit.

All the motion of translation of the fluid represented by the terms χ' disappears when the electro-motive force forms closed circuits; that is, when the components of the electro-motive force satisfy the equation of continuity throughout space. The energy of the fluid will then be merely that of the vortex filaments all through it; or, in other words, the energy will be entirely magnetic. It is evident without calculation, therefore, that this kind of fluid explains the magnetic attraction of currents without further hypothesis, and that the equation of Ampère, or some other giving the same results for closed circuits, could be readily deduced.

The principal advantage of this theory over that of Maxwell is in the idea of electro-motive force here introduced, by which an ordinary perfect fluid becomes available instead of one having a complicated and unknown law

connecting its molecules. But I do not yet attempt to explain electrostatic action.

The idea of electro-motive force here introduced has thus rotation as its basis. And by this idea we have thus done away with the necessity of any action similar to viscosity in the medium, but have shown that a force producing vortices at any one point will fill all space with vortices, which will be packed together according to the laws indicated by the equations.

I do not consider this theory as final, by any means, but only as one link in the chain, the first three links of which have been added by Thomson, Helmholtz, and Maxwell, and which *may* finally end in the true theory.

IV. *Extension of Equations to include the newly discovered Electro-magnetic Action.*

Leaving this theory, however tempting further discussion might be, and considering the equations again merely from an electrical point of view, we have now to inquire how the recently discovered electro-magnetic action will affect them. The following theory may be regarded as only preliminary.

Evidently we must add to the electro-motive forces which produce the original currents others depending on the current and the magnetic force at each point. These new electro-motive forces have been found to be linear functions of the magnetic force and the current at each point. At present they appear to be at right angles to the plane containing these two, but for the sake of generality I will first suppose the function to be general. Let the new electro-motive forces be A'', B'', and C'', and let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ be four new constants depending on the material. Then we can write evidently

$$A'' = \mathfrak{a}a\sqrt{b'^2+c'^2} + \mathfrak{b}a'\sqrt{b^2+c^2} + \mathfrak{c}\,(bc'-b'c) + \mathfrak{d}aa'$$
$$B'' = \mathfrak{a}b\sqrt{a'^2+c'^2} + \mathfrak{b}b'\sqrt{a^2+c^2} + \mathfrak{c}\,(ca'-c'a) + \mathfrak{d}bb'$$
$$C'' = \mathfrak{a}c\sqrt{a'^2+b'^2} + \mathfrak{b}c'\sqrt{a^2+b^2} + \mathfrak{c}\,(ab'-a'b) + \mathfrak{d}cc'.$$

Of these four constants, Mr. Hall has proved the existence of $\mathfrak{c}$, and also that $\mathfrak{a}$ is very small or zero. As to the other two constants, it would seem almost an impossibility to test their value with accuracy; Mr. Hall has attempted to do so by trying the resistance of a wire and also of a piece of gold leaf under magnetic action, but has as yet determined only that the effect is almost inappreciable on the resistance.

But if we suppose the action to be rotatory, as I pointed out in the last number of this journal, we can get a probable relation between $\mathfrak{c}$ and $\mathfrak{b}$. For let

$b' = c = a = a' = 0$, and we have the case of a current, c', and a magnetic force, b, and the equations reduce to

$$A'' = \mathfrak{c} b c'$$
$$B'' = \mathfrak{a} b c' = 0$$
$$C'' = \mathfrak{b} b c'.$$

If the action is a rotation of the current element around the line of force, we should have this rotation proportional to b and $\mathfrak{b}$ would be to $\mathfrak{c}$ in the ratio of the tangent of the angle of rotation to unity. But as the angle must be very minute, $\mathfrak{b}$ must be also very small compared with unity.

I have attempted to apply the conservation of energy to the case, but have not yet obtained any very good results, especially as conduction in a moving medium does not seem to be perfectly understood.

Let us then suppose all the constants except $\mathfrak{c}$ to be zero, and we then have

$$A'' = \mathfrak{c}\,(bc' - b'c)$$
$$B'' = \mathfrak{c}\,(ca' - c'a)$$
$$C'' = \mathfrak{c}\,(ab' - a'b).$$

The general equations for the electro-magnetic action alone will then be

$$F = 4\pi\mu\mu' \left\{ \iiint \left[A' + \mathfrak{c}\,(b_1 c' - b' c_1) \right] \frac{1}{r}\, dx dy dz - \frac{d\chi'}{dx} \right\}$$

$$G = 4\pi\mu\mu' \left\{ \iiint \left[B' + \mathfrak{c}\,(c_1 a' - c' a_1) \right] \frac{1}{r}\, dx dy dz - \frac{d\chi'}{dy} \right\}$$

$$H = 4\pi\mu\mu' \left\{ \iiint \left[C' + \mathfrak{c}\,(a_1 b' - a' b_1) \right] \frac{1}{r}\, dx dy dz - \frac{d\chi'}{dz} \right\},$$

where a_1, b_1, and c_1 indicate the *total* components of the magnetism due to magnets as well as electric currents, and a, b, and c are the components due to the currents alone.

F, G, and H do not include the vector potentials of the magnets.

$$a = \frac{dH}{dy} - \frac{dG}{dz} \qquad\qquad 4\pi\mu a' = \frac{dc}{dy} - \frac{db}{dz}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \qquad\qquad 4\pi\mu b' = \frac{da}{dz} - \frac{dc}{dx}$$

$$c = \frac{dG}{dx} - \frac{dF}{dy} \qquad\qquad 4\pi\mu c' = \frac{db}{dx} - \frac{da}{dy}.$$

If we wish F, G, and H to satisfy the equation of continuity, then the value of χ' will be, as we have found before,

$$\chi' = \frac{1}{2}\iiint \frac{1}{r}\left\{[A'+\mathfrak{c}(b_1c'-b'c_1)][x-x']+[B'+\mathfrak{c}(c_1a'-c'a_1)][y-y']+[C'+\mathfrak{c}(a_1b'-a'b_1)][z-z']\right\}dxdydz.$$

Otherwise we can give it any value we please, as it vanishes from the other equations. In case we make it zero, we shall have *

$$4\pi\mu a' = -\Delta^2 F$$
$$4\pi\mu b' = -\Delta^2 G$$
$$4\pi\mu c' = -\Delta^2 H.$$

The exact calculation will in general be very complicated. But as $\mathfrak{c}$ is very small in all substances so far experimented on, we can easily calculate the effect as a correction to the quantities calculated on the ordinary theory. But where the effect is to be calculated in a limited body, it then, in general, becomes very complicated. However, the solution is possible by ordinary integration for a thin circular disc or for a sphere, as we can then apply the method of images.

In some cases no currents will be produced in the body, but simply a difference of potential which can be measured by a galvanometer, as in Mr. Hall's experiment.

V. *Explanation of the Magnetic Rotation of the Plane of Polarization of Light.*

To apply these results to Maxwell's theory of light, we must assume that the same action which takes place in conductors with reference to conducted currents, also takes place in dielectrics with reference to displacement currents. It is almost impossible to detect this action experimentally, but we shall here follow out the consequence of its existence. I shall follow the method of Art. 783 of Maxwell's "Treatise," with the addition of this new action.

Assume at once $C=0$, $\psi=0$, and $J=0$ as they are afterwards taken or proved to be.

Let P, Q, and R be the components of the electro-motive forces acting at any point. The electro-motive force will be composed of two parts: first, the rate of variation of the vector potential as on the old theory; and, second, a term

* I use the expression Δ^2 to signify the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$, while Maxwell uses it in his *Theory of Light* with the opposite sign.

depending on the new action, and whose components we have designated by A'', B'', and C''. Adding these together, we have

$$P = -\frac{dF}{dt} + \mathfrak{c}\,(b_1 c' - b' c_1)$$

$$Q = -\frac{dG}{dt} + \mathfrak{c}\,(c_1 a' - c' a_1)$$

$$R = -\frac{dH}{dt} + \mathfrak{c}\,(a_1 b' - a' b_1).$$

The displacement currents a', b', and c' will be

$$a' = \frac{K}{4\pi}\frac{dP}{dt}$$

$$b' = \frac{K}{4\pi}\frac{dQ}{dt}$$

$$c' = \frac{K}{4\pi}\frac{dR}{dt},$$

and they are also expressed by the equations

$$4\pi\mu a' = -\Delta^2 F$$

$$4\pi\mu b' = -\Delta^2 G$$

$$4\pi\mu c' = -\Delta^2 H.$$

Hence we have by elimination

$$K\mu\left\{\frac{d^2F}{dt^2} - \frac{d}{dt}\mathfrak{c}\,(b_1 c' - b' c_1)\right\} - \Delta^2 F = 0$$

$$K\mu\left\{\frac{d^2G}{dt^2} - \frac{d}{dt}\mathfrak{c}\,(c_1 a' - c' a_1)\right\} - \Delta^2 G = 0$$

$$K\mu\left\{\frac{d^2H}{dt^2} - \frac{d}{dt}\mathfrak{c}\,(a_1 b' - a' b_1)\right\} - \Delta^2 H = 0.$$

Before the solution of these equations, of course the values of a_1, b_1, c_1 and a', b', c' must be substituted in terms of F, G, and H.

Let us now take the case of a plane polarized ray passing in the direction of the axis of z, with a magnetic force, c_1, along the same axis. The magnetic forces a, b, c, the variations of which constitute the waves of light, are very small; for Maxwell has calculated that in strong sunlight the maximum is about one-tenth of the horizontal intensity of the earth's magnetism. Hence we can write

$$K\mu\left\{\frac{d^2F}{dt^2}+\mathfrak{c}\frac{d}{dt}(b'c_1)\right\}-\frac{d^2F}{dz^2}=0$$

$$K\mu\left\{\frac{d^2G}{dt^2}-\mathfrak{c}\frac{d}{dt}(a'c_1)\right\}-\frac{d^2G}{dz^2}=0$$

$$K\mu\frac{d^2H}{dt^2}=0\,;$$

and, replacing b' and a' by their values, we have

$$K\mu\left\{\frac{d^2F}{dt^2}-\frac{\mathfrak{c}c_1}{4\pi\mu}\frac{d^3G}{dtdz^2}\right\}-\frac{d^2F}{dz^2}=0$$

$$K\mu\left\{\frac{d^2G}{dt^2}+\frac{\mathfrak{c}c_1}{4\pi\mu}\frac{d^3F}{dtdz^3}\right\}-\frac{d^2G}{dz^2}=0.$$

From the form of the equations we can well suppose that one solution is

$$F=r\cos(nt-qz)\cos mt$$

$$G=r\cos(nt-qz)\sin mt,$$

and making the substitution we find

$$\left\{K\mu(n^2+m^2)-q^2\left(1+\mathfrak{c}\frac{mc_1K}{4\pi}\right)\right\}\cos(nt-qz)\cos mt$$

$$-Kn\left\{2m\mu-\mathfrak{c}\frac{c_1q^2}{4\pi}\right\}\sin(nt-qz)\sin mt=0$$

$$\left\{K\mu(n^2+m^2)-q^2\left(1+\mathfrak{c}\frac{mc_1K}{4\pi}\right)\right\}\cos(nt-qz)\sin mt$$

$$+Kn\left\{2m\mu-\mathfrak{c}\frac{c_1q^2}{4\pi}\right\}\sin(nt-qz)\cos mt=0.$$

These are satisfied if we make the coefficients zero.

If V is the velocity in general of light in the medium, and V_0 the velocity in vacuo without magnetic action; if i is the index of refraction of the medium, and λ the complete wave length in the medium, and λ_0 in vacuo; we thus find

$$m=\mathfrak{c}\frac{\pi c_1}{2\mu\lambda^2}$$

$$V=\frac{n}{q}=\sqrt{\frac{1+\mathfrak{c}\frac{Kmc_1}{8\pi}}{K\mu}}\,.$$

$$V=\frac{1}{\sqrt{K\mu}}\left(1+\frac{K\mu m^2\lambda^2}{8\pi^2}\right).$$

These equations indicate that when a ray of plane polarized light passes in the direction of the lines of magnetic force, the plane of polarization will be rotated in a direction depending on the sign of the quantity $\mathfrak{C}$, which is the well-known action of Faraday. But the second expression which gives the velocity, and consequently the index of refraction, also depends on $\mathfrak{C}$, and thus indicates an acceleration of the velocity which is unknown. But this action is so very minute that it can probably never be measured.

If D is the length of the substance, the total angle of rotation of the beam will evidently be

$$\theta = m\frac{D}{V} = \mathfrak{C}\frac{\pi}{2\mu V_0}\frac{i^3}{\lambda_0^2}Dc_1\,.$$

This solution is rigorously exact for all cases where the index of refraction is not a function of the wave length. To get the value where the index varies, we can use the principle of the superposition of small quantities. Every plane polarized ray can be supposed to be made up of two circularly polarized rays; and to say that the plane of polarization is rotated simply means that one of the circularly polarized rays travels faster than the other; when one ray gains λ on the other the plane of polarization is rotated through the angle 2π. Hence, if V is the velocity of one and V' of the other, we have

$$V = V'\left(1 + \frac{\lambda}{D'}\right),$$

where D' is the distance in which the plane of polarization is rotated through the angle 2π.

But this effect will be augmented by the dispersion of the body, seeing that the velocity affects the wave length, and hence the index of refraction will be different for the two components. This further action can be taken into account by multiplying $\frac{V}{V'}$ by $\frac{i'}{i}$, and we then have

$$\frac{V}{V'}\frac{i'}{i} = 1 + \frac{\lambda}{D''}\cdot$$

So that D' has been changed to

$$D'' = \frac{\lambda}{\frac{i'}{i}\left(1 + \frac{\lambda}{D'}\right) - 1}\cdot$$

This can be put into the form

$$\frac{D''}{D'} = \frac{1}{\frac{i'}{i}\left\{1 + \frac{D'}{\lambda}\frac{i'-i}{i'}\right\}}\cdot$$

But
$$\frac{\lambda}{\lambda'}=\frac{D'+\lambda}{D'},$$

whence $\lambda-\lambda'=\frac{\lambda^2}{D'}$ and $\frac{i'-i}{i}=\frac{\lambda'-\lambda}{i}\frac{di}{d\lambda}+\text{etc.}$

Hence, omitting all quantities of the second order of smallness, we can write

$$\frac{D''}{D'}=\frac{1}{1-\frac{\lambda}{i}\frac{di}{d\lambda}};$$

and the angle of rotation, θ, will become

$$\theta=2\pi\frac{D}{D''}=\mathfrak{c}Dc_1\,.\frac{\pi}{2\mu V_0}\frac{i^2}{\lambda_0^2}\left(i-\lambda\frac{di}{d\lambda}\right),$$

which is of the same form as Maxwell's expression. Now Maxwell's equation is obtained from considerations entirely different from any which I have used in this paper. In obtaining them, Maxwell made no assumption as to the kind of motion which constitutes light, but merely assumed that the magnetic lines of force were vortices, and that the motion of the vortices caused a rotation of the motion constituting light. In my theory I have used no hypothesis as to the nature of magnetic force, but have simply calculated, from the known laws of magnetism and electricity, the action in this case according to Maxwell's theory of light. And the conclusion which we draw is that *the effect discovered by Mr. Hall is the same, or due to the same cause, as the rotation of the plane of polarization of light.*

It is interesting to repeat here the comparison made by Verdet between the various formulæ and observation.

The formulæ of Maxwell and Rowland, of Airy, and of Neumann are

$$\text{(I)}\qquad \theta=M\frac{i^2}{\lambda^2}\left(i-\lambda\frac{di}{d\lambda}\right)Dc_1$$

$$\text{(II)}\qquad \theta=M\frac{1}{\lambda^2}\left(i-\lambda\frac{di}{d\lambda}\right)Dc_1$$

$$\text{(III)}\qquad \theta=M\left(i-\lambda\frac{di}{d\lambda}\right)Dc_1.$$

The comparison of these formulæ with the experiments of Verdet* are as follows:

* Verdet, *Œuvres,* Vol. I. p. 262.

Bisulphide of Carbon.

	C	D	E	F	G
Observed rotation,	0.592	0.768	1.000	1.234	1.704
Calculated, formula I,	0.589	0.760	1.000	1.234	1.713
" " II,	0.606	0.772	1.000	1.216	1.640
" " III,	0.943	0.967	1.000	1.034	1.091

Creosote.

	C	D	E	F	G
Observed rotation,	0.573	0.758	1.000	1.241	1.723
Calculated, formula I,	0.617	0.780	1.000	1.210	1.603
" " II,	0.627	0.789	1.000	1.200	1.565
" " III,	0.976	0.993	1.000	1.017	1.041

To examine the direction of the action, we must see what the relative direction of the currents and magnetism are in the equations, as I have not taken the signs with respect to any system.

Let the positive direction of the current be the direction in which the positive electricity moves, and the positive direction of the magnetic lines of force be the direction in which the north pole tends to move. Then we easily find that our equations are on the right-handed screw system, the right-handed screw being such that if we turn it in the direction of the hands of a watch with its face toward us, it will move away from us. According to this system, Mr. Hall has found that the value of $\mathfrak{C}$ is positive for gold and some other diamagnetic substances, and negative for iron. Hence a magnetic force in the positive direction will cause the ray to be rotated in the positive direction in diamagnetic substances, and in the negative direction in magnetic ones, which is exactly what has been observed.

To compare the numerical amount of the revolution with observation, we can take the constants as observed by Mr. Hall for gold, and thus find at least whether it is of the proper order of magnitude.

From more recent observations than those published, Mr. Hall finds that, in the field of his magnet, he can cause the lateral electro-motive force to be at least as great as $\frac{1}{2000}$ of the force along the strip. According to the system of units used in this paper, the new electro-motive force will be in the case of conduction, the current passing along V and the magnetism being in the direction of z,

$$A'' = -\mathfrak{C}c_1 b' = -4\pi\mu'\mathfrak{C}c_1 B' ;$$

but Mr. Hall finds

$$\frac{B'}{A''} = -\ 2000 \text{ nearly.}$$

Hence, using the C. G. S. system, in which $\mu' = 2000$ nearly, we shall have

$$\mathfrak{c}c_1 = \frac{1}{4\pi} \text{ nearly for gold.}$$

The length of the substance in which the ray is rotated a complete revolution, or 360°, will then be

$$D = \frac{2\pi V_0}{mi} = \frac{4\ \mu\lambda_0^2 V_0}{\mathfrak{c}c_1^2\left(i - \lambda\frac{di}{d\lambda}\right)},$$

where it is to be noted that λ_0 is the length of a *complete* wave. Taking the wave of $\frac{1}{10000}$ cm. length, and the index of refraction 4, we find, supposing $\frac{di}{d\lambda} = 0$,

$$D = 240 \text{ cm. nearly.}$$

We do not know the magnetic force used by Verdet, but it was evidently of the same order of magnitude. He found D to be about as follows: 300 for heavy glass, 700 for flint glass. Hence the rotation calculated for gold is of the same order of magnitude as the rotation observed in some common substances.

Thus the new electro-magnetic phenomenon explains in the most perfect manner the magnetic rotation of the plane of polarization of light, and we are almost in the position to pronounce positively that the two phenomena are the same. Should this preliminary theory of the subject stand the test of time, it hardly seems to me that we can regard it in any other light than a demonstration of the truth of Maxwell's theory of light; for the rotation of the plane of polarization is thus a necessary consequence of the laws of electro-magnetism, and this, added to the other facts of the case, raises Maxwell's theory almost to the realm of fact.